Nonuniversality of compact support probability distributions in random matrix theory

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The two-point resolvent is calculated in the large-n limit for the generalized fixed and bounded trace ensembles. It is shown to disagree with that of the canonical Gaussian ensemble by a nonuniversal part that is given explicitly for all monomial potentials $V(M)=M^{2p}$. Moreover, we prove that for the generalized fixed and bounded trace ensemble all k-point resolvents agree in the large-n limit, despite their nonuniversality. [S1063-651X(99)10411-2]

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I. INTRODUCTION

Restricted trace ensembles (RTEs), introduced a long time ago in [1], are interesting for a couple of reasons. They possess compact support not only for infinite but also for finite n, where n is the size of the matrix. In the canonical ensemble equation (1.2) large values of matrix elements are only exponentially suppressed, whereas in the RTEs a sharp cutoff is introduced. For this reason the latter can be regarded as the corresponding microcanonical ensemble.

Much of the relevance of random matrix theory is related to universality properties of connected correlators in the large-n limit, that is, their independence of the details of the probability density that defines the matrix ensemble. The most famous property is the limiting form of the connected density-density correlator at ''short distances,'' also called the ''sine law.'' Very interesting also is a global universality property: it was found that smoothed connected correlators may be expressed by the same universal function. The original derivation made use of loop equations [2]; it was later rediscovered by diagrammatical expansion [3]. All derivations are valid for canonical ensembles with an arbitrary polynomial; therefore it was generally believed that this global universality property holds for all probability densities invariant under unitary transformations. In this paper we investigate two major questions, namely whether the RTEs also possess universal global correlations, which are independent of the details of the distribution, and, second, whether they are equivalent to the universality classes of the canonical ensemble. Notice that usual techniques, such as orthogonal polynomials, fail for RTEs because of the additional constraint on the matrix trace.

In order to address the above problems, in a previous publication [4] we introduced the following generalization of the RTEs:

$$\mathcal{P}_\phi(M)\equiv\frac{1}{\mathcal{Z}_\phi}\phi\left(A^2-\frac{1}{n}\mathrm{Tr}\,V(M)\right),\quad V(M)=\sum_{l=1}^{p} g_{2l}M^{2l},$$

$$\mathcal{Z}_\phi\equiv\int \mathcal{D}M\,\phi\left(A^2-\frac{1}{n}\mathrm{Tr}\,V(M)\right), \tag{1.1}$$

where $\phi(x)=\delta(x)$ or $\theta(x)$; and compared them to the canonical ensemble

$$\mathcal{P}(M)\equiv\frac{1}{\mathcal{Z}}\exp[-ng\,\mathrm{Tr}\,V(M)],$$

$$\mathcal{Z}\equiv\int \mathcal{D}M\,\exp[-ng\,\mathrm{Tr}\,V(M)]. \tag{1.2}$$

We have calculated the spectral density $\rho(\lambda)=\langle 1/n\,\mathrm{Tr}\,\delta(\lambda-M)\rangle$ of RTEs, which is equivalent to the one-point resolvent $G(z)=\langle 1/n\,\mathrm{Tr}(z-M)^{-1}\rangle$. Comparing it to the canonical ensemble we have shown that in the large-n limit they agree, provided that the scale factor g takes a well defined value determined by the values of the couplings g_{2l} in the potential $V(M)$ and by A^2. This holds despite the well known fact that the spectral density itself is nonuniversal. From the factorization property of correlators at large n we then concluded that all finite moments of the three ensembles coincide. The question now is whether this equivalence holds also for the connected part of higher correlation functions and thus for higher orders in $1/n$. Therefore in this paper we investigate all k-point correlators. We start with $k=2$:

$$G_\phi(z,w)\equiv\left\langle\frac{1}{n}\mathrm{Tr}\frac{1}{z-M}\,\frac{1}{n}\mathrm{Tr}\frac{1}{w-M}\right\rangle_\phi-\left\langle\frac{1}{n}\mathrm{Tr}\frac{1}{z-M}\right\rangle_\phi\left\langle\frac{1}{n}\mathrm{Tr}\frac{1}{w-M}\right\rangle_\phi, \tag{1.3}$$

where the subscript ϕ indicates the corresponding average. Here we have subtracted the factorized part. The two-point correlator as well as all higher k-point correlators are known to be universal for the canonical ensemble [2]. There, the subtraction corresponds to taking into account only connected diagrams of random surfaces. The corresponding connected density-density correlators can be obtained by taking the appropriate imaginary part, as given for example in [5]. (In contrast to Ref. [5] we define the k-point resolvents without a factor of n^{2k-2}.)

II. NONUNIVERSALITY OF $G_\phi(z,w)$

In the following we shall restrict ourselves to purely monomial potentials $V(M)=M^{2p}$. Since we want to show that the correlator $G_\phi(z,w)$ is *nonuniversal*, in principle only two examples of different potentials leading to a different correlator equation (1.3) would be sufficient. In a first step we will show that all expectation values of products of matrices have a $1/n^2$-expansion for the RTEs. When relating the averages to the corresponding canonical ones we can explicitly extract their expansion coefficients, which enables us to calculate $G_\phi(z,w)$.

It is useful to introduce the following representation for the δ and θ function:

$$\phi(x)=\int_{-\infty}^{\infty}\frac{dy}{2\pi}\frac{e^{(iy+\epsilon)x}}{(iy+\epsilon)^s},\quad \begin{cases} s=0, & \phi(x)=\delta(x),\\ s=1, & \phi(x)=\theta(x),\end{cases} \tag{2.1}$$

where $\epsilon=0^+$ is a small and harmless regulator that makes it possible to interchange integrals. Next we calculate the matrix integral

$$I_\phi^{\{k\}}(n,A)\equiv\int \mathcal{D}M\,\phi\left(A^2-\frac{1}{n}\mathrm{Tr}\,M^{2p}\right)\times M_{\alpha_1\beta_1}M_{\alpha_2\beta_2}\ldots M_{\alpha_k\beta_k}, \tag{2.2}$$

where the superscript $\{k\}$ summarizes the dependence on all the matrix indices. The volume element of the Hermitian $(n\times n)$ matrices $\Pi_i dM_{ii}\Pi_{i<j}d\,\mathrm{Re}\,M_{ij}\Pi_{i<j}d\,\mathrm{Im}\,M_{ij}$ is the usual product of independent entries. In the particular case $k=0$, Eq. (2.2) is just the partition function $\mathcal{Z}_\phi$. By inserting the representation equation (2.1) into Eq. (2.2) and exchanging the order of integrations, we exhibit that Eq. (2.2) is actually proportional to the analogous integral with canonical measure. Indeed,

$$I_\phi^{\{k\}}(n,A)=\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{(iy+\epsilon)A^2}\frac{1}{(iy+\epsilon)^s}\times\int \mathcal{D}M e^{-[(iy+\epsilon)/n]\mathrm{Tr}\,M^{2p}}M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k}. \tag{2.3}$$

The explicit dependence of the matrix integral

$$\int \mathcal{D}M e^{-a\,\mathrm{Tr}\,M^{2p}}M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k} = a^{-(n^2+k)/2p}\int \mathcal{D}M e^{-\mathrm{Tr}\,M^{2p}}M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k} \tag{2.4}$$

on the real positive parameter a, allows analytic continuation in the whole complex plane, cut along the negative part of the real axis. This provides a definition for Eq. (2.3), and we obtain

$$\begin{aligned} I_\phi^{\{k\}}(n,A)&=\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{(iy+\epsilon)A^2}\frac{1}{(iy+\epsilon)^s}\left(\frac{gn^2}{iy+\epsilon}\right)^{(n^2+k)/2p}\\ &\quad\times\int \mathcal{D}M e^{-ng\,\mathrm{Tr}\,M^{2p}}M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k}\\ &=(gn^2A^2)^{(n^2+k)/2p}\frac{(A^2)^{s-1}}{\Gamma\left(\frac{n^2+k}{2p}+s\right)}\\ &\quad\times\int \mathcal{D}M e^{-ng\,\mathrm{Tr}\,M^{2p}}M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k}, \end{aligned} \tag{2.5}$$

where $g>0$, and in the second step we have used Hankel's contour integral for the Γ function [6]. As a consequence, we obtain the RTE average expressed by the canonical average

$$\langle M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k}\rangle_\phi=\frac{I_\phi^{\{k\}}(n,A)}{I_\phi^{\{0\}}(n,A)} = s_{n,k}(g,A)\langle M_{\alpha_1\beta_1}\ldots M_{\alpha_k\beta_k}\rangle, \tag{2.6}$$

where

$$s_{n,k}(g,A)=(gn^2A^2)^{k/2p}\frac{\Gamma\left(\frac{n^2}{2p}+s\right)}{\Gamma\left(\frac{n^2+k}{2p}+s\right)}. \tag{2.7}$$

On the right-hand side (rhs) of Eq. (2.6) we average with respect to the canonical measure Eq. (1.2) for $V(M)=gM^{2p}$. The exact relation (2.6) may be exploited to relate the parameters of the RTEs to the parameters of the canonical model, so that at leading order in the large-n limit all moments of the form (2.6) are identical. It is, however, impossible to relate the parameters to obtain that the scaling factor $s_{n,k}(g,A)$ is unity up to $O(1/n^4)$. Indeed, if we assume

$$A^2=\frac{1}{2pg}+\frac{x}{n^2}+O\left(\frac{1}{n^4}\right) \tag{2.8}$$

and use the relation for ratios of Γ functions [6], we obtain

$$\begin{aligned} s_{n,k}(g,A)&=(2pgA^2)^{k/2p}\left[1-\frac{k}{2n^2}\left(\frac{k}{2p}+2s-1\right)+O\left(\frac{1}{n^4}\right)\right]\\ &=1+\frac{k}{n^2}\left(gx-\frac{k}{4p}-s+\frac{1}{2}\right)+O\left(\frac{1}{n^4}\right). \end{aligned} \tag{2.9}$$

This shows that, with the general relation (2.8), the $1/n^2$ expansion of the canonical measure translates into a $1/n^2$ expansion for the RTEs. (Using Stirling's formula one can easily convince oneself that $z^{b-a}[\Gamma(z+a)/\Gamma(z+b)]$ has an expansion in $1/z$.) The nonvanishing contribution at order $1/n^2$ in the scaling factor $s_{n,k}(g,A)$ with the k^2 dependence will be shown to lead to the nonuniversality of connected correlators.

The relation between the coefficients $c^{(j)}_{\{k\}}$ of the topological $1/n^2$ expansion in the canonical ensemble is simply related to the corresponding coefficients $d^{(j)}_{\{k\}}$ for the RTEs,

$$\langle M_{\alpha_1\beta_1}\dots M_{\alpha_k\beta_k}\rangle=\sum_{j=0}^{\infty} c^{(j)}_{\{k\}}\frac{1}{n^{2j}},$$

$$\langle M_{\alpha_1\beta_1}\dots M_{\alpha_k\beta_k}\rangle_\phi=\sum_{j=0}^{\infty} d^{(j)}_{\{k\}}\frac{1}{n^{2j}} \tag{2.10}$$

through Eq. (2.9)

$$d^{(0)}_{\{k\}}=c^{(0)}_{\{k\}},$$

$$d^{(1)}_{\{k\}}=c^{(1)}_{\{k\}}+k\left(gx-\frac{k}{4p}-s+\frac{1}{2}\right)c^{(0)}_{\{k\}}, \tag{2.11}$$

where we recall that we have $s=0,1$ for $\phi=\delta,\theta$, and the subscript $\{k\}$ summarizes all matrix indices. Equation (2.11) immediately implies the identity of the one-point resolvents [4]

$$G_\phi(z)=\frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\langle \mathrm{Tr}\, M^k\rangle_\phi$$
$$=\frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\left[d^{(0)}_{\{k\}}+O\left(\frac{1}{n^2}\right)\right]\overset{n\to\infty}{\to} G(z), \tag{2.12}$$

which has been shown in [4] for a larger class of potentials. Note that $G_\phi(z)$ is of order 1 since $d^{(0)}_{\{k\}}$ contains a power of n from the trace.

Next we turn to the two-point resolvent $G_\phi(z,w)$. Inserting Eq. (2.11) into the definition (1.3) we obtain

$$G_\phi(z,w)=\frac{1}{n^2}\sum_{k,l=0}^{\infty}\frac{1}{z^{k+1}w^{l+1}}(\langle \mathrm{Tr}\, M^k\,\mathrm{Tr}\, M^l\rangle_\phi-\langle \mathrm{Tr}\, M^k\rangle_\phi\langle \mathrm{Tr}\, M^l\rangle_\phi)$$
$$=\frac{1}{n^2}\sum_{k,l=0}^{\infty}\frac{1}{z^{k+1}w^{l+1}}\left\{d^{(0)}_{\{k,l\}}+d^{(1)}_{\{k,l\}}\frac{1}{n^2}-\left[d^{(0)}_{\{k\}}+d^{(1)}_{\{k\}}\frac{1}{n^2}\right]\left[d^{(0)}_{\{l\}}+d^{(1)}_{\{l\}}\frac{1}{n^2}\right]+O\left(\frac{1}{n^4}\right)\right\}$$
$$=\frac{1}{n^4}\sum_{k,l=0}^{\infty}\frac{1}{z^{k+1}w^{l+1}}\left[c^{(1)}_{\{k,l\}}-c^{(0)}_{\{k\}}c^{(1)}_{\{l\}}-c^{(1)}_{\{k\}}c^{(0)}_{\{l\}}-\frac{kl}{2p}c^{(0)}_{\{k\}}c^{(0)}_{\{l\}}+O\left(\frac{1}{n^2}\right)\right]. \tag{2.13}$$

Here we have made use of the fact that

$$d^{(0)}_{\{k,l\}}=c^{(0)}_{\{k,l\}}=c^{(0)}_{\{k\}}c^{(0)}_{\{l\}},$$

$$d^{(1)}_{\{k,l\}}=c^{(1)}_{\{k,l\}}+(k+l)\left(gx-\frac{k+l}{4p}-s+\frac{1}{2}\right)c^{(0)}_{\{k,l\}}. \tag{2.14}$$

As a consequence, the leading terms in $G_\phi(z,w)$ cancel as they should, leaving the remaining part of order $1/n^2$ when counting properly factors of n from the traces. Remarkably the result (2.13) does not depend upon the values of x and s. The first three terms in the last line of Eq. (2.13) give precisely the universal two-point resolvent of the canonical ensemble. The last term is new and can be written as a product of derivatives of the one-point resolvent $G(z)(=G_\phi(z))$. The final result reads

$$n^2G_\phi(z,w)\overset{n\to\infty}{\to} n^2G(z,w)-\frac{1}{2p}\partial_z(zG(z))\partial_w(wG(w)), \tag{2.15}$$

which holds for all monomial potentials $V(M)=gM^{2p}$, both $\phi=\delta,\theta$ and A^2 given by Eq. (2.8). Note that all terms in Eq. (2.15) are of order 1. The first term is the well known universal two-point resolvent [2]

$$n^2G(z,w)=\frac{1}{2(z-w)^2}\left(\frac{zw-a^2}{\sqrt{(z^2-a^2)(w^2-a^2)}}-1\right), \tag{2.16}$$

where a denotes the support of the eigenvalues $[-a,a]$. The notion of universality means that Eq. (2.16) is the same for any given polynomial potential sharing the same support [2]. We only have to assume that the couplings g_{2l} are such that the support is one arc. This is true in particular for the monomial potentials with $g>0$. The second term in Eq. (2.15), however, is *nonuniversal*, as the one-point resolvent itself is nonuniversal. Let us give two examples, the Gaussian and the purely quartic potential,

$$V(M)=gM^2:\quad G(z)=g(z-\sqrt{z^2-a^2}),\quad a^2=\frac{2}{g},$$

$$V(M)=gM^4:\quad G(z)=g(2z^3-(2z^2+a^2)\sqrt{z^2-a^2}),$$

$$a^4=\frac{4}{3g}. \tag{2.17}$$

Although in this case we have a potential depending only on one parameter g, which is thus in one to one correspondence with the end point of support a, the two resolvents in Eq.

(2.17) are different *functions* of z. Inserting them into the result for $G_\phi(z,w)$, Eq. (2.15), we obtain for $V(M)=gM^2$:

$$n^2G_\phi(z,w)=n^2G(z,w)-g^2\left(2z-\frac{2z^2-a^2}{\sqrt{z^2-a^2}}\right)\left(2w-\frac{2w^2-a^2}{\sqrt{w^2-a^2}}\right),$$

and for $V(M)=gM^4$:

$$n^2G_\phi(z,w)=n^2G(z,w)-g^2\left(8z^3-\frac{8z^4-4a^2z^2-a^4}{\sqrt{z^2-a^2}}\right)\times\left(8w^3-\frac{8w^4-4a^2w^2-a^4}{\sqrt{w^2-a^2}}\right), \tag{2.18}$$

where $n^2G(z,w)$ is given in Eq. (2.16) and A^2 in Eq. (2.8). These examples clearly demonstrate the nonuniversality of $G_\phi(z,w)$, which cannot be repaired by a suitable parameter redefinition.

Let us finally mention that we have verified Eq. (2.15) for the quadratic potential following an entirely different approach. As has already been emphasized in [4] the fixed trace ensemble $\phi=\delta$ can be obtained from the ''trace squared ensemble''

$$\mathcal{P}_l(M)\equiv\frac{1}{\mathcal{Z}_l}\exp(-l\{-2nA^2\,\mathrm{Tr}\,V(M)+[\mathrm{Tr}\,V(M)]^2\}),$$

$$\mathcal{Z}_l\equiv\int\mathcal{D}M\exp(-l\{-2nA^2\,\mathrm{Tr}\,V(M)+[\mathrm{Tr}\,V(M)]^2\}), \tag{2.19}$$

when taking the limit $l\rightarrow\infty$. The trace square terms add so-called ''touching'' interactions to the triangulated surface [7]. This representation of the fixed trace ensemble not only provides us with a different technical tool to check Eq. (2.15) for $p=1$, which we do not display here, it also gives us a diagrammatical interpretation in terms of Feynman graphs, which explains the existence of the $1/n^2$ expansion in Eq. (2.10) for the Gaussian fixed trace ensemble as a topological expansion.

It seems remarkable that the second term in Eq. (2.15), in the case of Gaussian resolvent $G_\phi(z)\rightarrow G(z)=(2/a^2)(z-\sqrt{z^2-a^2})$, has the same form of the analogous term that appears in the connected correlator for Wigner ensembles [8,9], written in different but equivalent forms, since

$$\partial_z(zG(z))=\frac{-2a^2[G(z)]^3}{4-a^2[G(z)]^2}=\frac{a^2}{4}\partial_z[G(z)]^2.$$

III. EQUIVALENCE OF ALL HIGHER-POINT RESOLVENTS OF THE RTEs

The two-point resolvent of the fixed and bounded trace ensemble has turned out to be identical in the large-n limit although nonuniversal. It is therefore natural to ask whether this equivalence also holds for all higher k-point resolvents. In the following we will show that this is indeed the case. Let us define the two generating functionals

$$Z_\phi[J]\equiv\sum_{k=0}^{\infty}\frac{1}{k!}\int dz_1\ldots dz_k\times\left\langle\frac{1}{n}\mathrm{Tr}\frac{1}{z_1-M}\ldots\frac{1}{n}\mathrm{Tr}\frac{1}{z_k-M}\right\rangle_\phi J(z_1)\ldots J(z_k), \tag{3.1}$$

$$W_\phi[J]\equiv\sum_{k=0}^{\infty}\frac{1}{k!}\int dz_1\ldots dz_k G_\phi(z_1,\ldots,z_k)J(z_1)\ldots J(z_k), \tag{3.2}$$

where

$$G_\phi(z_1,\ldots,z_k)\equiv\left\langle\frac{1}{n}\mathrm{Tr}\frac{1}{z_1-M}\ldots\frac{1}{n}\mathrm{Tr}\frac{1}{z_k-M}\right\rangle_\phi-\sum_{\sigma\in P}G_\phi(z_{\sigma(1)},\ldots,z_{\sigma(l_1)})\ldots G_\phi(z_{\sigma(l_{k-1}+1)},\ldots,z_{\sigma(k)}) \tag{3.3}$$

is the k-point resolvent. The sum runs over all different partitions P of the k arguments and thus over all different combinations of one- to $(k-1)$-point resolvents within total k indices. In the canonical ensemble this subtraction corresponds to taking only connected graphs into account. For this reason the k-point resolvent is there of the order $1/n^{2k-2}$. From field theory we know that the following relation between the generating functionals holds:

$$Z_\phi[J]=e^{W_\phi[J]}. \tag{3.4}$$

The correlators can be obtained in the usual way

$$\frac{\delta^k}{\delta J^k}\left\{\begin{array}{l}Z_\phi[J]\\ W_\phi[J]\end{array}\right|_{J=0}=\left\{\begin{array}{l}\left\langle\frac{1}{n}\mathrm{Tr}\frac{1}{z_1-M}\ldots\frac{1}{n}\mathrm{Tr}\frac{1}{z_k-M}\right\rangle_\phi\\ G_\phi(z_1,\ldots,z_k).\end{array}\right. \tag{3.5}$$

In [4] it has been shown that the ensemble averages of the fixed and bounded trace ensemble can be related:

$$\langle\mathcal{O}(M)\rangle_\delta=(1+c_n\partial_{A^2})\langle\mathcal{O}(M)\rangle_\theta, \tag{3.6}$$

where we have

$$c_n=2pA^2\frac{1}{n^2}\quad\text{for}\quad V(M)=M^{2p}. \tag{3.7}$$

Consequently the same relation holds for the generating functionals $Z_\phi[J]$ following their definition (3.1),

$$Z_\delta[J]=(1+c_n\partial_{A^2})Z_\theta[J]. \tag{3.8}$$

Using the relation (3.4) we can translate this into the generating functional for the resolvent operators

$$e^{W_\delta[J]}=[1+c_n(\partial_{A^2}W_\theta[J])]e^{W_\theta[J]}, \tag{3.9}$$

or equivalently,

$$W_{\delta}[J]=W_{\theta}[J]+\sum_{l=1}^{\infty}(-)^{l+1}\frac{1}{l}(c_n\partial_{A^2}W_{\theta}[J])^l, \tag{3.10}$$

where we have expanded the logarithm. Taking the functional derivative $\delta^k/\delta J^k$ and setting $J=0$ will truncate the infinite sum for the following reason. From the definition we have $W_{\phi}[J=0]=1$ and thus $\partial_{A^2}W_{\phi}[J=0]=0$. For this reason only terms will persist where at least one functional derivative $\delta/\delta J$ acts on $\partial_{A^2}W_{\phi}[J]$. We finally obtain

$$\begin{aligned}G_{\delta}(z_1,\ldots,z_k)=&(1+c_n\partial_{A^2})G_{\theta}(z_1,\ldots,z_k)\\&+\sum_{\sigma\in P;l=2,\ldots,k}(-)^{l+1}\frac{1}{l}\\&\times[c_n\partial_{A^2}G_{\theta}(z_{\sigma(1)},\ldots,z_{\sigma(l_1)})]\ldots\\&[c_n\partial_{A^2}G_{\theta}(z_{\sigma(l_{k-1}+1)},\ldots,z_{\sigma(k)})].\end{aligned} \tag{3.11}$$

Here the sum runs again over all partitions P of the k arguments and l counts the number of blocks or resolvents into which the arguments are divided. To prove the desired equivalence between the two k-point resolvents we need to know the order in $1/n^2$ of all terms on the rhs. Following the diagrammatic approach mentioned at the end of the last section, where the fixed trace ensemble is represented by the trace squared one, we obtain the same counting of powers as in the canonical ensemble already mentioned,

$$G_{\delta}(z_1,\ldots,z_k)=O\left(\frac{1}{n^{2k-2}}\right), \tag{3.12}$$

at least in the Gaussian case. In the following we will assume that same holds for the monomial potentials. We have checked this explicitly for the two- and three-point resolvent using the definition (3.3) and the relation (2.11). It now follows easily by induction that Eq. (3.12) also holds for the bounded trace ensemble and that we have

$$n^{2k-2}G_{\theta}(z_1,\ldots,z_k)\overset{n\to\infty}{\to}n^{2k-2}G_{\delta}(z_z,\ldots,z_k), \tag{3.13}$$

which generalizes Eq. (2.13) for the two-point resolvents. Namely in Eq. (3.11) on the rhs the second term in the first line is obviously subleading, due to $c_n\sim 1/n^2$. Using induction in the sum, each term is of $O(n^{-(2l_1+2(l_2-l_1)+\cdots+2(k-l_{k-1})})=O(n^{-2k})$, which is also subleading.

In the above derivation no explicit use has been made of the δ or θ measure apart from the fact that $\theta'(x)=\delta(x)$. Instead of this, we could have used, for example, $\phi(x)=x\theta(x)$ and $\phi(x)=\theta(x)$ because of $[x\theta(x)]'=\theta(x)$. More generally we can extend the proof of relation (3.13) to an infinite class of RTEs with

$$\phi(x)=\left\{\delta(x),\theta(x),\left(\frac{1}{j!}x^j\theta(x)\right)_{j\in\mathrm{N}_+}\right\}, \tag{3.14}$$

showing that all their k-point resolvents are equivalent at large n. We only need to show the starting point for $k=2$ since we have used induction. This can be shown as follows. When we calculate the matrix integral $I_{\phi}^{\{k\}}(n,A)$ in Eq. (2.2), we allow the parameter s in the representation (2.1) to take all non-negative integer values, which is then a representation for all the measures introduced in Eq. (3.14). The same derivation goes through up to the result for the two-point resolvent Eq. (2.15), as we have kept s general and explicit everywhere.

Let us conclude this section with a final remark. In Ref. [10] a topological expansion was introduced and calculated for each resolvent,

$$G(z_1,\ldots,z_k)=\sum_{h=0}^{\infty}\frac{1}{n^{2h}}G_h(z_1,\ldots,z_k). \tag{3.15}$$

If we introduce the same expansion here for the $G_{\phi}(z_1,\ldots,z_k)$, a short look at relation (3.11) tells us that already for $h=1$ (''genus 1'') the equivalence (3.13) breaks down:

$$n^{2k-2+2h}G_{h\geq1,\theta}(z_1,\ldots,z_k)\neq n^{2k-2+2h}G_{h\geq1,\delta}(z_1,\ldots,z_k). \tag{3.16}$$

In this sense we have shown that only the ''planar'' ($h=0$) k-point resolvents of the fixed and extended bounded trace ensembles agree.

IV. CONCLUSIONS

We have proved the nonuniversality of the two-point resolvent $G_{\phi}(z,w)$ of the generalized fixed and bounded trace ensembles by comparing it to the universal two-point resolvent of the canonical ensemble. Apart from the general results for $G_{\phi}(z,w)$ for all monomial potentials $V(M)=M^{2p}$ we have explicitly displayed its nonuniversal parts in two examples, the quadratic and the pure quartic potential. Furthermore, we have extended the equivalence of the generalized fixed and generalized bounded ensemble in the large-n limit from all finite moments [4] to all k-point resolvents, which probe higher orders in $1/n^2$.

While we have shown that global universality fails for the generalized RTEs, the issue of universality of correlators at short distance, possibly matching with the canonical ensemble, is still open. We plan to come back to this interesting question in the future.

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